Agglomerative percolation on the Bethe lattice and the triangular cactus

Huiseung Chae, Soon-Hyung Yook,* and Yup Kim†

Department of Physics and Research Institute for Basic Sciences, Kyung Hee University, Seoul 130-701, Korea

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Abstract

We study the agglomerative percolation (AP) models on the Bethe lattice and the triangular cactus to establish the exact mean-field theory for AP. Using the self-consistent simulation method, based on the exact self-consistent equation, we directly measure the order parameter P_∞ and average cluster size S. From the measured P_∞ and S we obtain the critical exponents β_k and γ_k for $k = 2$ and 3. Here β_k and γ_k are the critical exponents for P_∞ and S when the growth of clusters spontaneously breaks the Z_k symmetry of the k-partite graph [12]. The obtained values are $\beta_2 = 1.79(3)$, $\gamma_2 = 0.88(1)$, $\beta_3 = 1.35(5)$, and $\gamma_3 = 0.94(2)$. By comparing these values of exponents with those for ordinary percolation ($\beta_\infty = 1$ and $\gamma_\infty = 1$) we also find the inequalities between the exponents, as $\beta_\infty < \beta_3 < \beta_2$ and $\gamma_\infty > \gamma_3 > \gamma_2$. These results quantitatively verify the conjecture that the AP model belongs to a new universality class if Z_k symmetry is broken spontaneously, and the new universality class depends on k [Lau *et al.*, Phys. Rev. E **86**, 011118 (2012)] .

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* syook@khu.ac.kr

† Corresponding author:ykim@khu.ac.kr

I. INTRODUCTION

Percolation transition describes the emergence of large-scale connectivity [1]. It has been extensively studied in various branches of science due to its wide range of applications to many phenomena such as sol-gel transition and polymerization, resistor networks, and epidemic spreading [1]. The first theoretical model for the percolation is the random or ordinary percolation in which a vacant site or a vacant bond of the background lattice is randomly chosen to be occupied. The percolation transition in the random percolation is normally known to be continuous [1]. The percolation has been extensively studied during last 3 or 4 decades to be considered as a mature branch of sciences.

However anomalous physical properties of exotic models on the percolation recently triggered some new studies. One kind of studies [2] was on the explosive percolation, which was first known to show supposedly discontinuous transition on the complete graph (CG) [2, 3]. But subsequent studies on the explosive percolation have shown that the transition of the explosive percolation on CG or the mean-field transition is continuous [4–7].

Another kind of studies was on the agglomerative percolation (AP) [8–12]. In AP one cluster is randomly selected instead of a bond or a site. Then the selected cluster merges all the nearest neighboring clusters to form a new cluster. The phase transition in AP is shown to be continuous, but belongs to a new universality class different from the class of the random percolation if the base structure of AP is bipartite [12]. On the bipartite structure like a two-dimensional square lattice the merging process spontaneously breaks the Z_2 symmetry at the transition threshold, which is the origin of the new universality class [12]. In contrast, the universality of the transition of AP on the triangular lattice, which is not bipartite, is the same as that of the random percolation [12]. Using analytical methods and numerical simulations, APs on the one-dimensional ring [8], the two-dimensional square lattice and triangular lattice [9], critical tree [10], and complex network [11] were studied. Through these studies AP on bipartite graphs is shown to belong to a new universality class different from that of the random percolation.

To understand and establish a new universality class of the critical phenomena clearly and precisely the exact mean-field theory for the new model must be first understood. However the mean-field theory (MFT) for AP on bipartite networks was not clearly understood yet. To get MFT of AP, the analytic theory based on the generating function of the Erdös-Rényi

(ER) random network was attempted [11]. However this analytic approach predicted the critical exponent γ as $\gamma = 1/2$, but from the numerical simulation on ER graph $\gamma = 0.88(10)$ is obtained, which is significantly larger than $\gamma = 1/2$. This suggests a possibility that the analytic approach based on the generating function [11] is still far from completion. Furthermore ER graph is not exactly bipartite. The numerical simulation study on the exact bipartite random graph earned only the critical exponent ν and the fractal dimension D of the giant cluster as $\nu = 4.7(2)$ and $D = 0.567(6)$, which are close to those for ER network but differ by more than one standard deviation [12]. Therefore, at the present stage, MFT for AP on the bipartite graphs are far from completion.

Recently the complete graph is widely used as a testbed for MFT [2, 4–6]. However the complete graph is not bipartite and one growth step of AP on the complete graph makes the entire graph a new single cluster. In contrast the Bethe lattice (infinite homogeneous Cayley tree) is the exact bipartite graph on which AP can be well defined. Moreover, the Bethe lattice is physically a very important substrate or medium on which MFTs for various physical models become exact [14]. The analytic treatments of magnetic models [15], percolation [1, 14], localization [14], and diffusion [16] on the Bethe lattice give important physical insights to subsequent developments of the corresponding research fields. Therefore, if the critical phenomena of AP on the Bethe lattice is completely understood, one knows MFT for AP exactly.

One of the theoretical merits of the Bethe lattice is that one can set up exact self-consistent equations on the lattice. Recently we have developed an exact self-consistent simulation method for an arbitrary percolation process on the Bethe lattice [7]. From the self-consistent simulation method, we have shown that the phase transition of the Achlioptas-type explosive percolation [2] undergoes continuous transition regardless of the details of growth rules. In this paper we analyze the critical properties of AP on the Bethe lattice by use of the developed self-consistent simulation. In the self-consistent simulation the order parameter P_∞ and the average size S of finite cluster on the Bethe lattice are directly measured. Therefore the exponents β and γ are also obtained directly without the finite size scaling, and our work can indeed establish exact MFT of AP.

In addition Lau *et al.* suggested the modified AP on the k-partite graph, which we call AP_k [12]. So AP_2 means the original AP on the bipartite graph. Based on the simple arguments, the transition of AP_k is conjectured to belong to another new universality class

when the growth of clusters in AP_k spontaneously breaks the Z_k symmetry of the k-partite graph [12]. However the conjecture have never been confirmed quantitatively, yet. Therefore, in this paper we also study the MFT of AP_3 by using the triangular cactus structure, which is an expanded structure of the Bethe lattice and exactly tripartite [13, 14]. By the self-consistent simulation we will also find the mean-field exponents β and γ for AP_3, or β_3 and γ_3. Finally from the results of AP_2 from the Bethe lattice and AP_3 from the triangular cactus, the inequalities between β_2 (β for AP_2) and β_3 and between γ_2 (γ for AP_2) and γ_3 will be provided in the mean-field level. From the obtained inequalities, we will suggest the inequalities for all β_k's and γ_k's.

This paper is organized as follows. The ordinary AP or AP_2 on the Bethe lattice is studied based on the self-consistent simulation in Sec. II. AP_3 on the triangular cactus is defined and studied Sec. III. Finally we summarize our results in Sec. IV.

II. AP_2 ON THE BETHE LATTICE

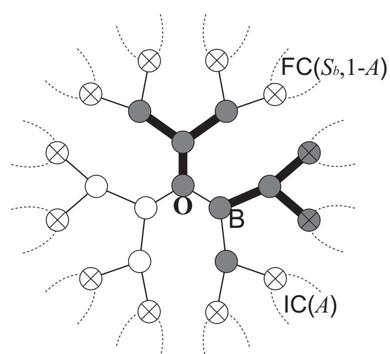

FIG. 1. Schematic diagram for AP_2 on the the Bethe lattice with $z = 3$. The center part consists of a three-generation Cayley tree with edge sites denoted by $\otimes$. Each edge site is connected to an infinite cluster (IC) with the probability A or to a finite cluster (FC) of average size of S_b with the probability $1 - A$. Thick lines mean occupied bonds and thin lines mean vacant bonds. If the cluster B is selected, gray sites merge into one cluster.

In the ordinary AP or AP_2 one cluster is randomly selected instead of a bond or a site and the cluster merges all the nearest neighboring clusters to form a new cluster. This means that in each growth process multiple bonds can be occupied at the same time. Therefore the natural control parameter in AP_2 is the number of clusters per site n instead of the fraction

p of the occupied bonds or sites [9, 12].

The Bethe lattice is the infinite Cayley tree in which tree structures connected to the center site $\mathbf{O}$ are identical to one another as schematically shown in Fig. 1 [1, 14]. The Bethe lattice is of course bipartite. Therefore AP_2 on the Bethe lattice is expected to belong to a new universality class different from that of the random percolation.

Let us now briefly explain the self-consistent simulation method for arbitrary percolation on the Bethe lattice with coordination number z in Ref. [7]. In the method we originally used the fraction p of occupied bonds or sites, but we use the number n of clusters per site in this paper. To set up self-consistent equations on which the simulation method is based, first consider a part of the Bethe lattice with m generations from the center site $\mathbf{O}$, which has total $N_0 = 1 + z(k^m - 1)/(k - 1)$ sites, where $k = z - 1$. To make a complete Bethe lattice, one should add an infinite branch to each of zk^{m-1} edge sites. To calculate the order parameter $P_\infty(n)$ of percolation, which is defined by the probability for $\mathbf{O}$ to belong to an infinite cluster at a given n, we need to know the probability A with which an occupied edge site connected to an infinite cluster. Let $P_{m\infty}(n, A)$ be P_∞ which is calculated from a Bethe lattice with the m generations from $\mathbf{O}$ and zk^{m-1} infinite branches. Then the self-consistent equation for P_∞ becomes

$$P_\infty = P_{m\infty}(n, A) = P_{m'\infty}(n, A) \tag{1}$$

for any combination of $\{m, m'\}$. Let us define $P_{mst}(n, A, S_b)$ as the probability that a cluster including $\mathbf{O}$ with s sites and t edge sites occurs within the m-generation tree. Then

$$P_{m\infty}(n, A) = 1 - \sum_t (1 - A)^t \sum_s P_{mst}(n, A, S_b) \tag{2}$$

where S_b is the average size of the finite cluster connected to an edge site of the m-generation tree as shown in Fig. 1. The self-consistent equation for the average size S of the finite clusters including $\mathbf{O}$ can also be written as

$$S = S_m(n, A, S_b) = S_{m'}(n, A, S_b) \tag{3}$$

where

$$S_m(n, A, S_b) = \frac{\sum_{s,t} P_{mst}(n, A, S_b)[s + tS_b](1 - A)^t}{1 - P_\infty}. \tag{4}$$

If one cannot calculate $P_{mst}(n, A, S_b)$ analytically to solve the self-consistent equations, one should estimate $P_{mst}(n, A, S_b)$ indirectly. One of such indirect methods is a simulation

method. We have developed a simulation method to solve self-consistent equations, which we call the self-consistent simulation [7]. In the self-consistent simulation, $P_{mst}(n, A, S_b)$ is estimated by the relation $P_{mst}(n, A, S_b) = N_{mst}(n, A, S_b)/N_{cluster}$, where $N_{mst}(n, A, S_b)$ is the number of clusters including **O** with s sites and t edge sites within the m-generation tree that occurred in simulations. Of course, $N_{cluster}$ is the total number of clusters which includes **O** within the m-generation tree that occurred in the same simulation runs. In the simulation both $P_{mst}(n, A, S_b)$ and $P_{m'st}(n, A, S_b)$ are estimated simultaneously using the Bethe lattice with the m-generation tree if $m > m'$.

Since we don't know A and S_b a priori, the iteration processes are needed in self-consistent simulation. From initially guessed values for A and S_b, the final or saturated values of A and S_b are obtained by the iteration of unit simulation process. The unit simulation process consists of the following two steps. (I) By use of the simulation runs based on the given values A and S_b, $P_{mst}(n, A, S_b)$ and $P_{m'st}(n, A, S_b)$ are estimated. (II) From the estimated $P_{mst}(n, A, S_b)$, new A and S_b are calculated by utilizing self-consistent equations (1) and (3). In the unit simulation process to get new A and S_b, the quantities like $P_{mst}(n, A, S_b)$ are estimated by averaging over at least 10^6 simulation runs. Such unit process is repeated until A and S_b reach the saturation values. Using the saturated values of A and S_b, P_∞ and S are estimated from Eqs. (2) and (4). In the self-consistent simulation, it should be careful to choose $m'(< m)$ for a given m as addressed in Ref. [7]. If m' is too small, the clusters within the m'-generation tree cannot have physical properties of AP_2 enough to give physically plausible solutions for self-consistent equations (1) and (3). If m' is very close to m, $P_{m\infty}(p, A)$ is numerically not so much distinct from $P_{m'\infty}(p, A)$ and the self-consistent equation (1) hardly gives the physically right solution. From the simulations with various sets of $\{m, m'\}$ it is confirmed that suitable choice of m' should be in the interval $m/3 < m' < m/2$.

The results of the self-consistent simulation with $z = 3$, $m = 20$, $m' = 9$ for AP_2 are displayed in Figs. 2 and 3. From the data for P_∞ in Fig. 2 the critical density n_c and the order parameter exponent β_2 are obtained based on the equation

$$P_\infty \simeq (n_c - n)^{\beta_2}, \tag{5}$$

which holds for the ordered phase or for $n < n_c$ near the critical point, i.e., $n \to n_c^-$. The obtained n_c and β_2 are $n_c = 0.7223(1)$ and $\beta_2 = 1.79(3)$. We have checked the results for the

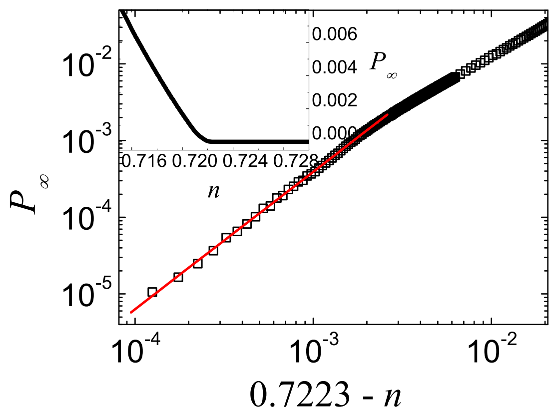

FIG. 2. (Color online) Plot of P_∞ for AP_2 against $n_c - n$ with $n_c = 0.7223$. The line denotes the relation (5) with $n_c = 0.7223$ and $\beta_2 = 1.79$. Inset is the raw-data plot of P_∞ against n.

simulation for some other combinations $(m = 14, m')$ with $m/3 < m' < m/2$ and found the same results. For another consistent checks we also applied the self-consistent simulation on the Bethe lattice with $z = 6$ to obtain $\beta_2 = 1.79(3)$. These numerical results for β_2 is close to the previous estimate $\beta_2 = 1.78(8)$ on the ER graph [11], but our estimate has much smaller errors.

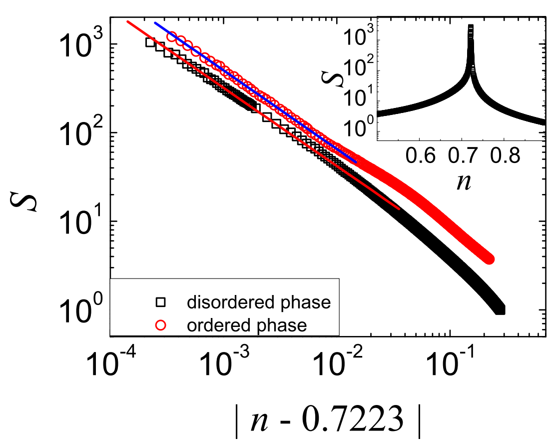

FIG. 3. (Color online) Plot of S for AP_2 against $|n_c - n|$ for ordered phase $(n < n_c)$ and disordered phase $(n > n_c)$. The lines denote the relations (6) with $\gamma_2^- = \gamma_2^+ = 0.88$ and $n_c = 0.7223$. Inset is the raw-data plot of S against n.

From the data for the average size S of finite clusters in Fig. 3 and the equation

$$S \simeq \begin{cases} |n-n_c|^{-\gamma_2^-} & \text{if } n_c < n \\ |n-n_c|^{-\gamma_2^+} & \text{if } n_c > n \end{cases}, \tag{6}$$

we also estimated n_c, γ_2^- and γ_2^+. Obtained n_c is nearly the same as that obtained from the data in Fig. 2. We also obtain $\gamma_2 = \gamma_2^- = \gamma_2^+ = 0.88(1)$, in which no asymmetry is found between the disordered phase ($n_c < n$) and the ordered phase $n_c > n$. The result $\gamma_2 = 0.88(1)$ is also consistent with the previous estimate $\gamma_2 = 0.88(10)$ on the ER Graph. We also obtained $\gamma_2 = 0.88(3)$ on the Bethe lattice with $z = 6$. These results clearly show that the obtained values of β_2 and γ_2 for AP_2 are significantly different from those for the random percolation [1].

In conclusion our estimates $\beta_2 = 1.79(3)$ and $\gamma_2 = 0.88(1)$ on the Bethe lattice are far more precise MFT exponents for AP_2 on the bipartite graph, since the dimensionality of the Bethe lattice is infinite.

III. AP_3 ON THE TRIANGULAR CACTUS

The triangular cactus was first introduced by Fisher and Essam [13] to investigate the effects of loops [14] on the percolation. As shown in Fig. 4 (a), the triangular cactus with coordination number $z = 4$ can be constructed from the Bethe lattice with $z = 3$. Each site in the Bethe lattice is replaced with a triangle of three sites to form the triangular cactus as shown in Fig. 4 (a). Thus the dimensionality of the triangular cactus is infinite as the Bethe lattice. Moreover, the triangular cactus is exactly tripartite, not bipartite as shown in Fig. 4 (b). Therefore one can expect that the critical phenomena of AP_2 on the triangular cactus belong to the random percolation universality class.

Recently, Lau *et al.* suggested the modified AP on the k-partite graph, which we called AP_k [12]. It is conjectured that the universality class of AP_k depends on k [12]. However AP_k for $k \geq 3$ has never been quantitatively studied, yet. In this section AP_3 on the triangular cactus is studied to obtain MFT of AP_3.

In AP_3 on the triangular cactus one cluster is randomly selected and the cluster merges some of the nearest neighboring clusters into a new cluster, instead of all neighboring clusters in AP_2 on bipartite graph. In a tripartite graph, initially, three colors are arranged such that no pair of nearest neighbor sites has the same color. Therefore we can identify the cluster

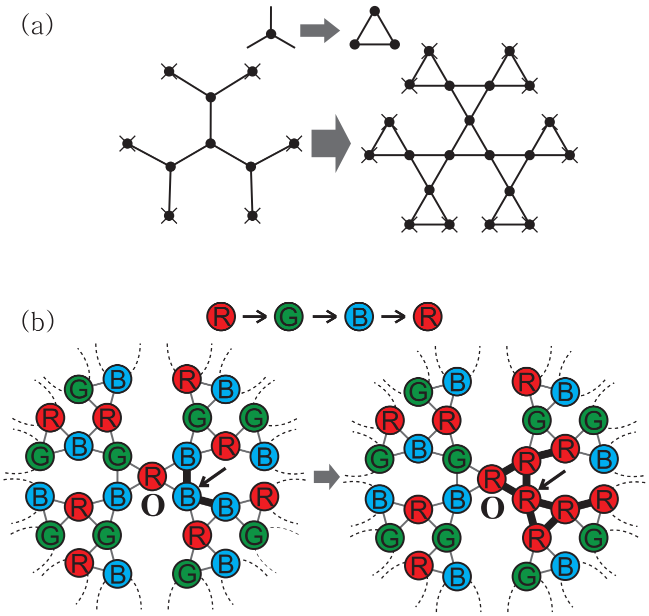

FIG. 4. (Color online) (a) Formation of the triangular cactus from the Bethe lattice with $z = 3$. In the cactus each site of the Bethe lattice is replaced with a triangle composed of three sites. Each edge site which is denoted by "X" is connected to an infinite branch. (b) AP_3 on the triangular cactus. The triangular cactus is tripartite as shown in the figure. If a blue-colored cluster indicated by an arrow is selected as in the left figure, it agglomerates all "R" neighbors and becomes a "R"-colored cluster by the rule, $R \to G \to B \to R$ as in the right figure.

by colors such as red (R), green (G), and blue (B). AP_3 is defined such that a selected cluster with "R" can join only with neighbors of the color "G", "G" can join only with neighbors of the color "B", "B" can join only with neighbors "R", based on a cyclic rule, $R \to G \to B \to R$ [12]. For example, a certain cluster with "R" is selected, then the cluster merges all neighboring clusters colored by "G" into a new cluster, and the merged cluster becomes a new "G"-colored cluster from the rule, $R \to G \to B \to R$. One can apply the $R \to B \to G \to R$ rule to the model, but it cannot be physically different from the model with the $R \to G \to B \to R$ rule.

For MFT of AP_3 on the tripartite graph, we use the self-consistent simulation method for AP_3 on the triangular cactus. The self-consistent simulation is almost the same as that for AP_2 on the Bethe lattice. First consider a part of the triangular cactus with m-generations from $\mathbf{O}$, which has total number of sites $N_0 = 2^{m+2} - 3$. To make a complete triangular cactus, one should add an infinite branch to each of 2^{m+1} edge sites. Other details of the

self-consistent simulation on the triangular cactus are exactly the same as those on the Bethe lattice. For instance, Eqs. (1)-(4) are the self-consistent equations not only for the Bethe lattice but for the triangular cactus.

We first checked the results of self-consistent simulation with $m = 20, m' = 9$ for AP on the triangular cactus. From the data for P_∞ and S, n_c, β and γ is estimated as $n_c = 0.6761(1)$, $\beta = 1.01(2)$, and $\gamma = 1.00(2)$. Since the triangular cactus is not bipartite but tripartite, this result supports that the critical phenomena of ordinary AP on the tripartite graph belong to the random percolation universality class with $\beta = \gamma = 1$ as expected in Ref. [12].

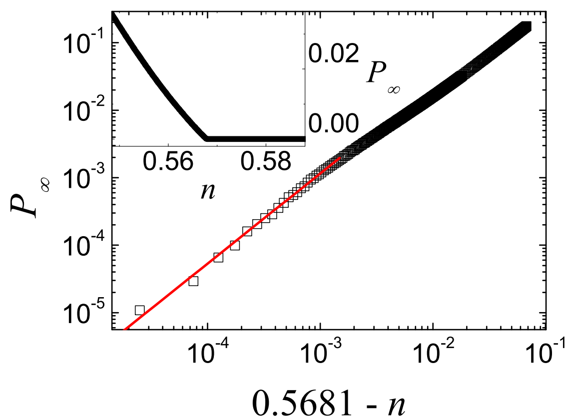

FIG. 5. (Color on line) Plot of P_∞ for AP_3 on the triangular cactus against $n_c - n$ with $n_c = 0.5681$. The line denotes the relation similar to (5) with $n_c = 0.7223$ and $\beta_3 = 1.35$. Inset is the raw-data plot of P_∞ against n.

On the other hand, the critical phenomena of AP_3 is different from AP on the triangular cactus. The results of the same self-consistent simulation for AP_3 on the triangular cactus are shown in Figs. 5 and 6. By using the similar equations to Eqs. (5) and (6) we have obtained the order parameter exponent β_3 and the susceptibility exponent γ_3 for AP_3 on the triangular cactus. The results are $\beta_3 = 1.35(5)$ and $\gamma_3 = 0.94(2)$ with $n_c = 0.5681(1)$ as shown in Figs. 5 and 6. In conclusion our estimated β_3 and γ_3 are the first MFT exponents for AP_3, since the dimensionality of the triangular cactus is infinite.

The obtained exponents β_3 and γ_3 satisfy the relations $\beta_\infty < \beta_3 < \beta_2$ and $\gamma_\infty > \gamma_3 > \gamma_2$, where $\beta_\infty(= 1)$ and $\gamma_\infty(= 1)$ are the MFT exponents of the random percolation. The

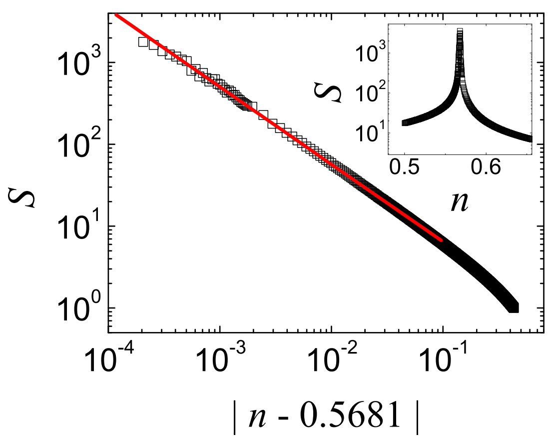

FIG. 6. (Color on line) Plot of S for AP_3 on the triangular cactus against $n_c - n$ with with $n_c = 0.5681$ for disordered phase ($n > n_c$). S for ordered phase ($n < n_c$) is not shown, because it behaves almost the same as that for disordered phase. The line denotes the relation similar to Eq. (6) with $\gamma_3^- = 0.94(2)$. Inset is the raw-data plot of S against n.

relations of exponents suggests that the MFT exponents of AP_k on the k-partite graph approach to those of the random percolation as k increases.

IV. SUMMARY

Finding the exact MFT is the first step to understand the various physical properties of a new model. AP was suggested as a natural extension of the standard percolation model. Some numerical studied for AP have been done on lower-dimensional lattices and random graphs [8, 10–12]. Based on those numerical studies, it was conjectured that AP belongs to a new universality class if the growth of cluster breaks Z_k symmetry on k-partited graph. However, the mean-field approach based on the evolutionary dynamics of clusters did not agree with the numerical simulations. This strongly indicates that AP is not fully understood even in mean-field level [11]. Therefore, in order to provide an exact MFT, we apply the self-consistent simulation method [7] to APs on the Bethe lattice and triangular cactus. From the direct and precise measurement of P_∞ and S through the self-consistent simulation, we obtain $\beta_2 = 1.79(3)$ and $\gamma_2 = 0.88(1)$ on the Bethe lattice when Z_2 symmetry is broken spontaneously at the transition threshold. Similarly, we obtain $\beta_3 = 1.35(5)$ and

$\gamma_3 = 0.94(2)$ on triangular cactus if Z_3 symmetry is broken spontaneously. However, since the triangular cactus is not bipartite, AP model on triangular cactus gives $\beta = 1.01(2)$ and $\gamma = 1.00(2)$. This result shows that ordinary AP on triangular cactus belongs to the same universality class with ordinary percolation. Therefore, the results for AP_3 on triangular cactus provide the exact MFT verifying the Lau *et al.*'s arguments [12]. In addition, by comparing the obtained critical exponents with those of random percolation, we also find the inequalities $\beta_\infty < \beta_3 < \beta_2$ and $\gamma_\infty > \gamma_3 > \gamma_2$. These inequalities also quantitatively verify the conjecture that the universality class of AP_k depends on k [12].

ACKNOWLEDGMENTS

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